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A NOTE ON STIRLING'S FORMULA FOR THE GAMMA FUNCTION

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Abstract. We present a new short proof of Stirling's formula for the Gamma function. Our approach is based on the Gauss product formula and on a remark concerning the existence of horizontal asymptotes.

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Stirling's formula is an approximation for large factorials, precisely,

$$n! \approx \sqrt{2\pi} n^{n+1/2} e^{-n} \tag{S}$$

in the sense that the ratio of the two sides tends to 1 as $n \to \infty$. This formula was discovered by Abraham de Moivre as part of his contribution to the normal approximation for the binomial distribution. His first derivation did not explicitly determine the constant $\sqrt{2\pi}$ but in a 1731 addendum he acknowledged that James Stirling was able to determine the constant, using Wallis' formula.

The literature concerning Stirling's formula is very large, counting hundreds of items on JSTOR. See for example [2], [3], [7], [8], [9] and [11].

The problem of extending the factorial to non-integer arguments was solved by Euler in 1729, by introducing the Gamma function via an infinite product, rewritten by Gauss under the form

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\cdots(x+n)}, \quad x > 0. \tag{Γ}$$

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In January 1730, Euler announced the integral representation of this function,

$$\Gamma(x) = \int_0^1 (-\ln s)^x \, ds,$$

which, via the change of variable $t = \ln s$, becomes the familiar Euler integral,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \text{for } x > 0,$$

Stirling's formula has a companion for the Gamma function,

$$\Gamma(x+1) \approx \sqrt{2\pi} x^{x+1/2} e^{-x} \quad \text{as } x \to \infty, \tag{SL}$$

first noticed by Pierre-Simon Laplace [6].

The aim of our note is to provide a short proof of the formula (SL), based on Euler's initial definition of the Gamma function and the following remark concerning the existence of horizontal asymptotes:

Lemma 1. *Suppose that* $(a_n)_n$ *is an increasing sequence of positive real numbers such that* $\sup a_n = \infty$ *and* $\sup (a_{n+1} - a_n) < \infty$, *and* $F : [0, \infty) \to \mathbb{R}$ *is a differentiable function such that* $\lim_{n\to\infty} F(a_n) = \ell$ *and* $\lim_{x\to\infty} F'(x) = 0$.

Then $\lim_{x\to\infty} F(x) = \ell$.

Proof. According to our hypotheses, for $\varepsilon > 0$ arbitrarily fixed there is $\delta > 0$ such that $|F'(x)| \leq \varepsilon$ whenever $x \geq \delta$. Choose an index N such that $a_N \geq \delta$. Since every $x \geq a_N$ lies in an suitable interval $[a_n, a_{n+1})$, an appeal to Lagrange's Mean Value Theorem yields

$$\begin{aligned} |F(x) - \ell| &\leq |F(x) - F(a_n)| + |F(a_n) - \ell| \\ &\leq \varepsilon \sup_n (a_{n+1} - a_n) + |F(a_n) - \ell|, \end{aligned}$$

whence $\lim_{x\to\infty} F(x) = \ell$. □

Our proof of the formula (SL) makes use of the auxiliary function

$$F(x) = \ln\left(\frac{\Gamma(x+1)e^x}{x^{x+1/2}}\right), \quad x > 0.$$

According to Stirling's formula, $\lim_{n\to\infty} F(n) = \ln\left(\sqrt{2\pi}\right)$, and thus Lemma 1 reduces the proof of (SL) to the fact that $\lim_{x\to\infty} F'(x) = 0$. This can be done by noticing the *Weierstrass product formula*,

$$\begin{aligned} \Gamma(x) &= \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\cdots(x+n)} = \frac{1}{x} \lim_{n\to\infty} \frac{e^{x \ln n}}{\left(1 + \frac{x}{1}\right)\cdots\left(1 + \frac{x}{n}\right)} \\ &= \frac{1}{x} \lim_{n\to\infty} \frac{e^{x(\ln n - 1 - 1/2 - \cdots - 1/n)} e^{x + x/2 + \cdots + x/n}}{\left(1 + \frac{x}{1}\right)\cdots\left(1 + \frac{x}{n}\right)} \\ &= \frac{1}{x} e^{-\gamma x} \prod_{n=1}^\infty \left(1 + \frac{x}{n}\right)^{-1} e^{x/n}, \end{aligned} \tag{W}$$

where $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right)$ is Euler's constant. By taking the logarithm of both sides of the formula (W) we obtain

$$\ln\Gamma(x) = -\ln x - \gamma x + \sum\nolimits_{n=1}^{\infty}\left(\frac{x}{n} - \ln\left(1+\frac{x}{n}\right)\right),$$

whence

$$\begin{aligned}\frac{\Gamma'(x)}{\Gamma(x)} &= -\frac{1}{x} - \gamma + \sum\nolimits_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{x+n}\right)\\ &= -\gamma + \sum\nolimits_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{x+n-1}\right).\end{aligned} \qquad (DS)$$

Here we applied the classical theorem on term by term differentiation, which asks for the local uniform convergence of the series of derivatives. See Thomson, Bruckner and Bruckner [10], Corollary 9.35, p. 579. This convergence follows from the Weierstrass M-test because on every interval $(0, R]$ with $R > 0$, the general term of the series (DS) verifies

$$\left|\frac{1}{n} - \frac{1}{x+n}\right| \leq \frac{R}{n^2} \text{ for } x \in (0, R],$$

and the series $\sum_{n\geq 1}\frac{1}{n^2}$ is convergent.

Therefore

$$\begin{aligned}F'(x) &= \frac{\Gamma'(x+1)}{\Gamma(x+1)} - \ln x - \frac{1}{2x}\\ &= -\gamma - \frac{1}{x+1} + \sum\nolimits_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{x+n}\right) - \ln x - \frac{1}{2x}\\ &= \lim_{n\to\infty}\left(\ln\frac{n}{x} - \sum\nolimits_{k=0}^{n}\frac{1}{x+k+1}\right) - \frac{1}{2x}.\end{aligned}$$

Taking into account that

$$\ln(x+2+k) - \ln(x+1+k) < \frac{1}{x+k+1} < \ln(x+1+k) - \ln(x+k)$$

we obtain that

$$-\frac{1}{2x} \leq F'(x) \leq \ln\frac{x+1}{x} - \frac{1}{2x},$$

whence $\lim_{x\to\infty}(\ln F(x))' = 0$. The proof of the formula (SL) is done.

Gosper's algorithm [4] for acceleration the rate of convergence of series yields a better approximation to $n!$, precisely,

$$n! \approx \sqrt{\left(2n+\frac{1}{3}\right)\pi n^n e^{-n}}.$$

The same argument as above allows us to conclude that

$$\Gamma(x+1) \approx \sqrt{\left(2x+\frac{1}{3}\right)\pi} x^x e^{-x}.$$

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